Ordinary Double Points through Mirror Symmetry

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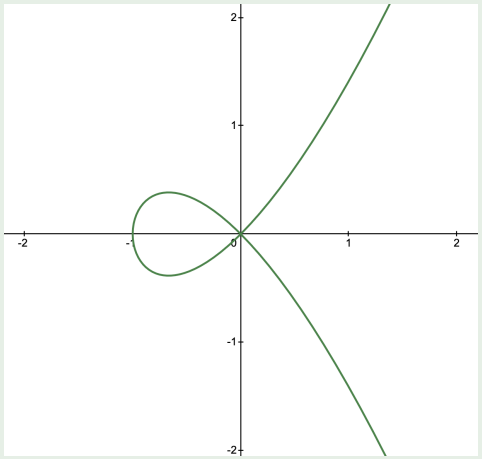
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Ordinary Double Points

- Let X be an algebraic variety given by $F(x_1, \ldots, x_n) = 0$.
- A point $p \in X$ is *singular* if $\frac{\partial F}{\partial x_1}(p) = \cdots = \frac{\partial F}{\partial x_n}(p) = 0$.
- An *ordinary double point* (ODP) is a singularity which can be locally described as $0 = x_1^2 + \cdots + x_n^2 +$ (higher order terms).

Example

Consider the curve $y^2 = x^2 + x^3$ on the right. The point (0,0) on the curve is an ODP.



The Maximum Number of ODPs

Motivating Question

For a polynomial of degree d in $\mathbb{P}^n(\mathbb{C})$, what is the maximum number of ODPs it can have?

- For $n = 2$, given by $\mu = \frac{(d-1)(d-2)}{2}$.

- For $n = 3$,

d	μ
1	0
2	1
3	4
4	16
5	31
6	65
≥ 7	???

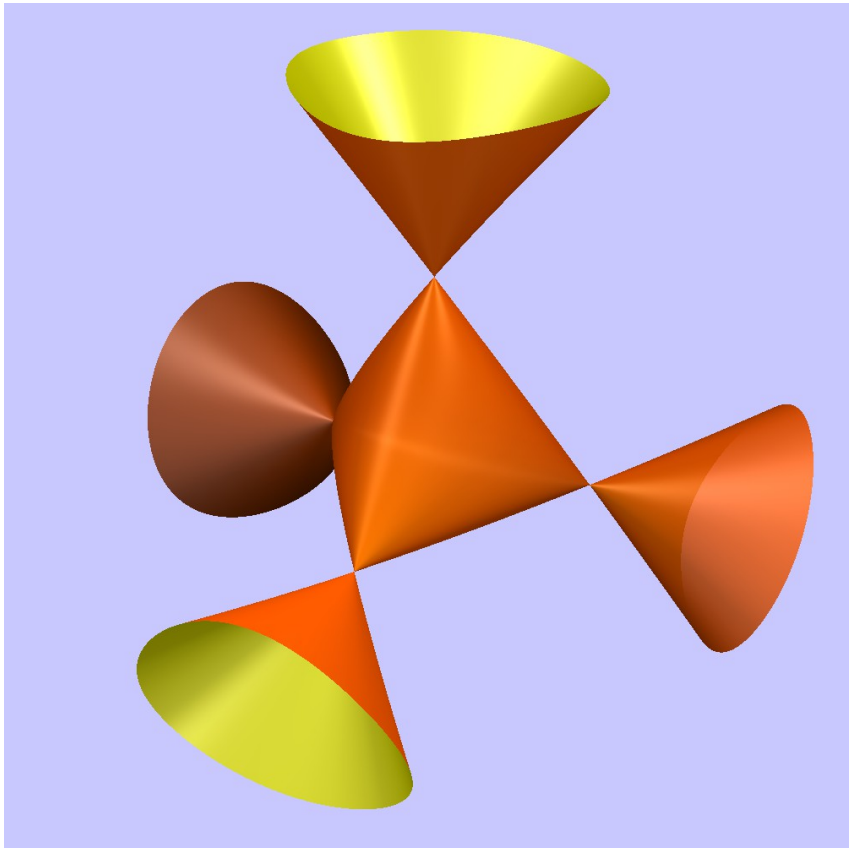
- For $n = 4$,

d	μ
1	0
2	1
3	10
4	45
≥ 5	???

Goal

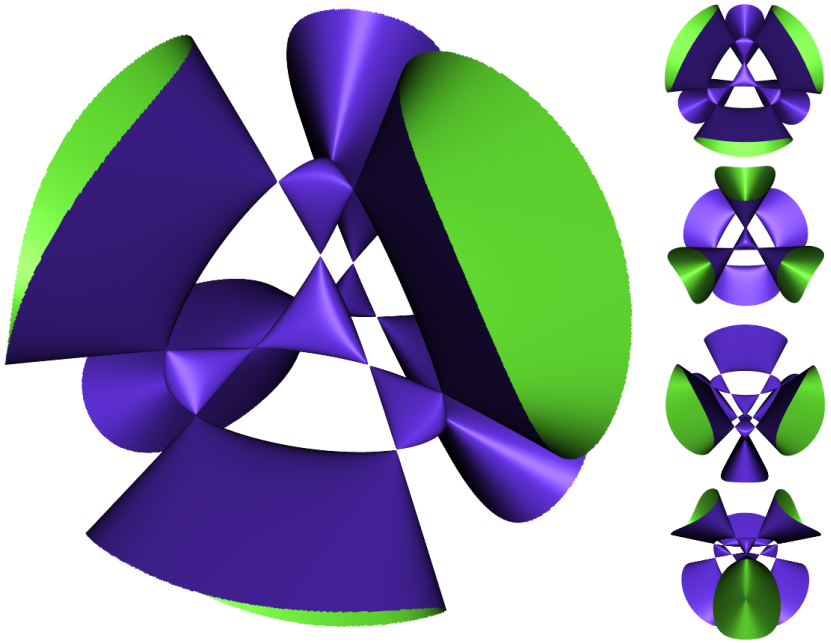
To reproduce these results using mirror symmetry, and to gain a deeper understanding of why these values are maximum.

The Maximal Varieties for $n = 3$



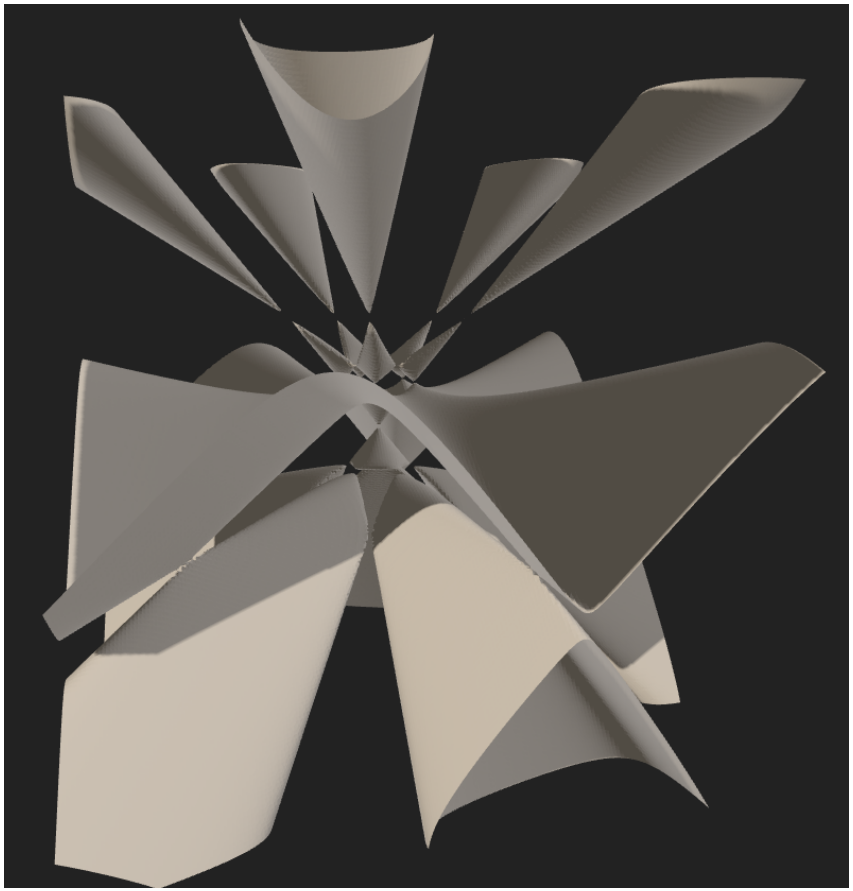
$d = 3$: Cayley Cubic with $\mu = 4$

The Maximal Varieties for $n = 3$



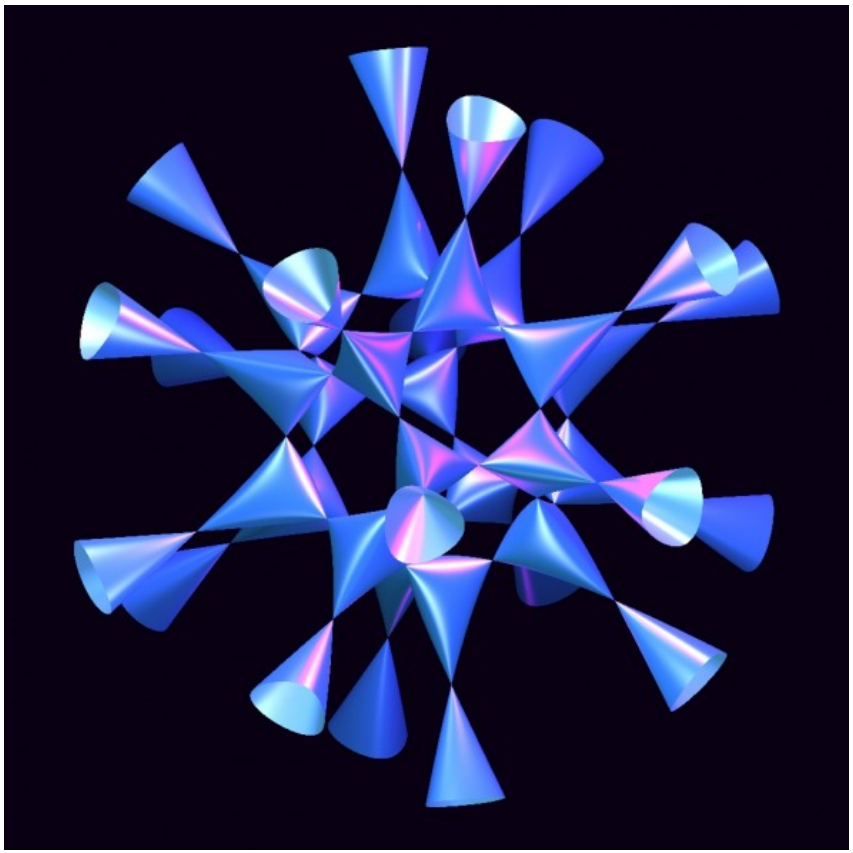
$d = 4$: Kummer Surface with $\mu = 16$

The Maximal Varieties for $n = 3$



$d = 5$: Togliatti Surface with $\mu = 31$

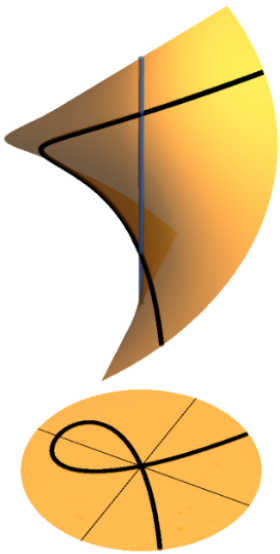
The Maximal Varieties for $n = 3$



$d = 6$: Barth Sextic with $\mu = 65$

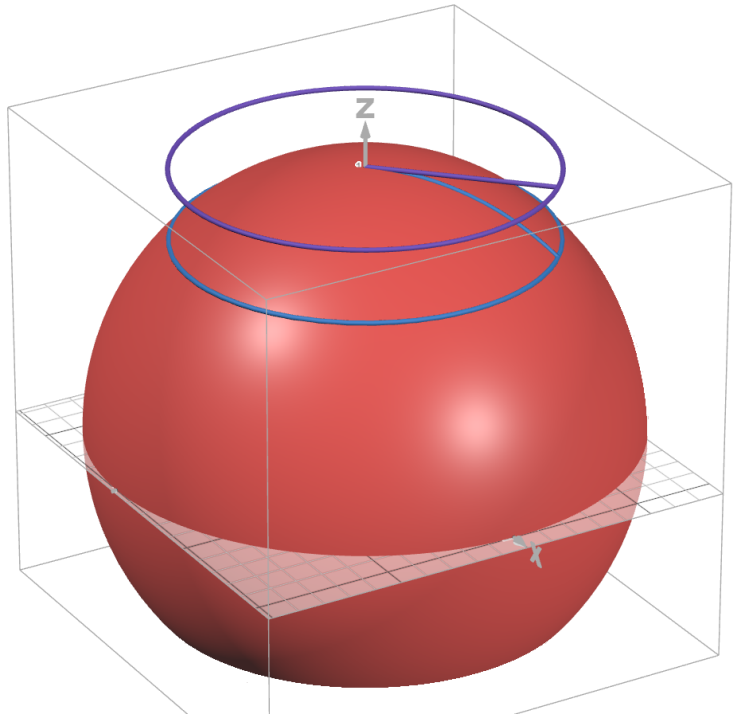
Degeneration and Blowups

- A *blowup* at a singular point $p \in X$ is a transformation where p is replaced by a parametrization of all lines passing through p.
- All singularities can be resolved (i.e. made smooth) through a finite number of blowups.
- Using the example of $y^2 = x^2 + x^3$, we have the following picture:



Fano Varieties

- Fano Varieties have positive curvature, which essentially means that the surface covered by the purple arc will be larger than the one by the blue one of the same length:



- For our cases a degree d variety $X \subseteq \mathbb{P}^n$ is Fano when $d \leq n+1$.

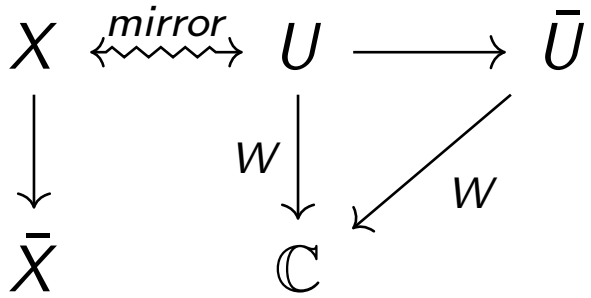
Mirror Symmetry

- It originated from physics, particularly from String Theory research.
- It allows us to investigate the mirror to gain insight onto the actual object we are interested in.
- It is called this because certain invariants are swapped in the mirror picture.

It is only proven for certain cases, and conjectured to hold for the others.
It has not been proven in our setting yet in full generality.

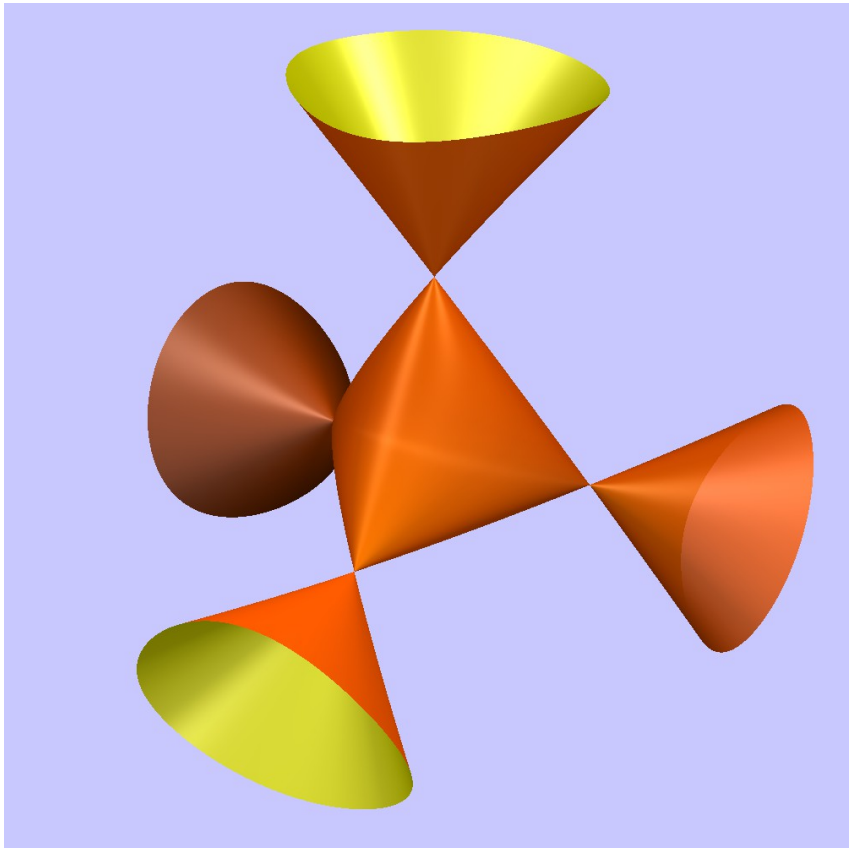
Mirror Symmetry for Fano Varieties

- The for a fano variety X, the mirror gives us a picture in the following manner:



- Here, $\bar{X}$ is the original surface we want to study with k ODPs.
- $\bar{U}$ is the degeneration of U with the maximum number of ODPs. We expect it to contain the same number as $\bar{X}$.
- In the mirror, we are studying the fibers: $W^{-1}(\lambda)$ for $\lambda \in \mathbb{C}$. These can be degenerated to obtain the ODPs.

The Cayley Cubic with $\mu = 4$



$d = 3$: Cayley Cubic with $\mu = 4$

Del Pezzo Surfaces

- A *Del Pezzo surface* is a Fano variety of dimension 2. These are fully classified and easy to work with.
- Here are our results:

Surface	d	μ	Mirror Fibrations	#ODPs from mirror
$\mathbb{P}^2$	9	0	$3 \cdot I_1$	0
$\mathbb{P}^1 \times \mathbb{P}^1$	8	1	$I_2 + 2 \cdot I_1$	1
$\mathrm{Bl}\mathbb{P}^2$	8	0	$4 \cdot I_1$	0
$\mathrm{Bl}^2\mathbb{P}^2$	7	1	$I_2 + 3 \cdot I_1$	1
$\mathrm{Bl}^3\mathbb{P}^2$	6	2	$I_3 + I_2 + I_1$	2
$\mathrm{Bl}^4\mathbb{P}^2$	5	2	$I_5 + 2 \cdot I_1$	2
$\mathrm{Bl}^5\mathbb{P}^2$	4	4	$I_1^* + I_1$	4
$\mathrm{Bl}^6\mathbb{P}^2$	3	4	$IV^* + I_1$	4
$\mathrm{Bl}^7\mathbb{P}^2$	2	6	$III^* + I_1$	5
$\mathrm{Bl}^8\mathbb{P}^2$	1	6	$II^* + I_1$	5

Extra Condition

Unfortunately the mirror construction fails for the case of degree 1 and 2. This is because the following condition is needed:
Let $\bar{X}$ be surface with maximum ODPs. We require there exists a hyperplane H such that

- H does not pass through any of $\bar{X}$'s singularities.
- The intersection $E = H \cap \bar{X}$ is singular.

Cases in $\mathbb{P}^4$

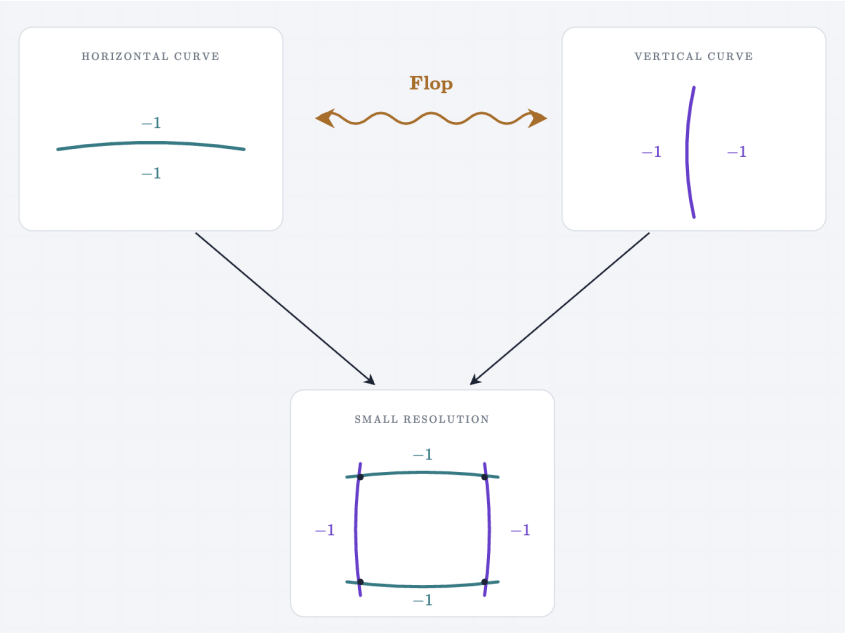
We then focused on the following cases:

d	μ
1	0
2	1
3	10
4	45
≥ 5	???

For $d = 3$ we obtained the expected result of 10 ODPs in the mirror picture, but the $d = 4$ case had an extra complication.

Flops

- A *flop* essentially "exchanges" one curve with another, with possibly different interactions with the surfaces of a variety.
- This is done by a *small resolution*, which is reached by blowing up the curve.
- From there, we can blow down into two different curves, representing the possibilities of the flop.



Flops Program

- Since need to consider all possible contractions from U to $\bar{U}$ in the mirror, we need to take flops into account, since they may give a U that may blow down to a $\bar{U}$ that has more ODPs.
- For this we made a program that computes these flops and searches through all possible configurations to select the highest μ, using tools from toric geometry.
- The example we will show with the program is the "Burkhardt Quartic", which is the case $n = d = 4$ on our table earlier. This has 45 ODPs.

Q&A